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## Report Title

A Mathematical Model of Network Communication

### ABSTRACT

The behavior of a communication network can be modeled as a flow of traffic units along links connected by nodes. We derive a node/link network model and connect it to a fluid-like model of traffic flow. The discrete node/link model emphasizes packet queuing and the flow of packets from spatial point to spatial point. The model assumes that packets reside in buffers at each node, and are classified by their destination and the length of time they have resided in the buffer.

An algorithm was created for packets to exit the buffer at each node according to their age and travel to the next node along a predetermined path to their destination. This algorithm calculates the rate at which packets distribute themselves to the next link in the route to their destination, assumes a source of packets originating at the node, and subtracts packets whose destination is that particular node. The continuum model derived from this discrete flow model leads to a flow continuity equation. The continuity equation describes the density of packets as a function of time and space, so that we are able to predict changes in global flow patterns and optimal paths in the network. Solutions to the flow equations in one dimension show that if the sources are too strong or the flow is restricted, the packet density grows at the nearest upstream node. When the source strength is reduced, or when flow is restored, the buffered packets flow at capacity until the density has been reduced.

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**List of papers submitted or published that acknowledge ARO support during this reporting period. List the papers, including journal references, in the following categories:**

**(a) Papers published in peer-reviewed journals (N/A for none)**

**Number of Papers published in peer-reviewed journals:**

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**Number of Papers published in non peer-reviewed journals:** 0.00

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### **(c) Presentations**

James R. Gatewood, Donald Drew

A Derivation of Spatial Internet Traffic Flow, 2005 The Walter Lincoln Hawkins Conference, Rensselaer Polytechnic Institute, April, 2005

James R. Gatewood, Donald Drew

Communication Network Flow Model

2008 World Congress in Computer Science, Computer Engineering and Applied Computing, July 2008, Las Vegas, NV

James R. Gatewood, Donald Drew

Communication Network Flow Model

2008 SIAM Annual Meeting, July 2008 San Diego, CA

James R. Gatewood, Donald A. Drew

A Mathematical Model of Communications Networks

2010 SIAM Annual Meeting, July 2010 Pittsburgh, PA

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**Non Peer-Reviewed Conference Proceeding publications (other than abstracts):**

**Number of Non Peer-Reviewed Conference Proceeding publications (other than abstracts):**

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Number of Manuscripts:

Number of Inventions:

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NAME	PERCENT SUPPORTED
James R. Gatewood	1.00
FTE Equivalent:	1.00
Total Number:	1

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NAME	PERCENT SUPPORTED
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Total Number:	

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NAME	PERCENT SUPPORTED	National Academy Member
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**Names of personnel receiving PHDs**

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James R. Gatewood
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**Sub Contractors (DD882)**

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Final Technical Report for ARO

**A Mathematical Model of Network Communication**

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## Abstract

The behavior of a communication network can be modeled as a flow of traffic units between nodes connected by links. We derive a node/link network model and connect it to a fluid-like model of traffic flow. The discrete node/link model emphasizes packet queuing and the flow of packets from spatial point to spatial point. The model assumes that packets reside in buffers at each node, and are classified by their destination and the length of time they have resided in the buffer. An algorithm was created for packets to exit the buffer at each node according to their age and travel to the next node along a predetermined path to their destination. This algorithm calculates the rate at which packets distribute themselves to the next link in the route to their destination, assumes a source of packets originating at the node, and subtracts packets whose destination is that particular node. The continuum model derived from this discrete flow model leads to a flow continuity equation. The continuity equation describes the density of packets as a function of time and space, so that we are able to predict changes in global flow patterns and optimal paths in the network. Solutions to the flow equations in one dimension show that if the sources are too strong or the flow is restricted, the packet density grows at the nearest upstream node. When the source strength is reduced, or when flow is restored, the buffered packets flow at capacity until the density has been reduced.

**Keywords:** Communication Networks, Flow Model, Packets, Flux, Continuity Equation

## 1 Introduction

Rapid communication is essential in many areas of endeavor. In everyday work and pleasure, we communicate important financial and personal information; in military situations, security and lives depend on it. Existence and functioning in the information age requires rapid and accurate transmission of data in wire, optical and wireless networks. Many systems can be modeled as networks where traffic units flow along links connecting nodes [24, 21, 22]. Modeling an aspect of flow on communication networks allows us to better understand the underlying dynamics of such networks, and particularly how the routing of units (packets) affects the gross features of the flow (delay, congestion) on the network [22].

A communication network is composed of links between nodes in a network that carries information from its source to its destination. Everything that travels on communication networks can be considered as “quanta” called packets. Email, web pages, instant messaging programs, voice communication, etc., are all broken down into chunks of information which travel throughout a communication network.

Packets are generated and delivered at host computers. A router forwards packets within a computer network along an “optimal path” to another router [1]. A packet might have to go through several routers before it reaches its destination. Communication networks are called packet-switching because routers

store and make forwarding decisions at the data and network layers to direct packets through the network.

In this model of network flow, network traffic is described mathematically by a link flow model for the buffering of packets and packet movement between discrete sets of nodes on links at discrete times. We shall use the discrete description and an associated service area to derive a continuum description for the density of packets (packets per square kilometer, for example) and flow (packets per second per kilometer of flow boundary) which creates a fluid-like model. Specifically, we model the spatial distribution of packet flow. We seek out the variables deemed most important that influence traffic flow; allowing us to create a simplified view of network communication, and concentrate on packet distribution as it depends on spatial location. For instance, the only information we need about a packet at some location in the network is its destination.

Modeling flow on a communication network as a fluid will allow us to (i) describe the spatial structure and evolution of normal and congested flow on large communication networks; (ii) predict changes in flow pattern due to changes in the spatial density of sources and per-link traffic and (iii) determine the optimal route a packet would take to get to its destination (possibly improving upon the existing algorithms).

On some conceptual scale, flow on a communication network appears to behave like a fluid flow. We propose a model fluid flow in terms of density of packets as a function of spatial coordinates and time. We suggest that this approach allows study of gross features of flow in the network under changes of conditions. This report is organized as follows: Section 2 makes an analogy to lattice gases and sets the structure for the flow model that we wish to describe. Section 3 describes a buffer model showing how the packets evolve in age and transition from buffered to the send queue. Section 4 describes the discrete communication network model. Section 5 explains the continuum model and how it was derived from the discrete model which leads to the continuity equation. Section 6 presents several solutions to the model illustrating how it behaves in situations of blockage and saturation.

## 2 Lattice Gas Communication Network Model

In this Section, we propose a model for the spatial distribution of packet flow and dynamics using the concepts of the Lattice Gas Cellular Automata model and the relation of lattice gases to flow equations. Packet motions in large communication networks behave somewhat like a lattice gas. In a network of mobile users, wireless communication is essential, but the spatial range of wireless is limited. Consequently, messages are routed from node to node, so that flow in all “directions” is possible.

A cellular automata model is a system composed of adjacent cells or sites (usually organized as a regular lattice) which evolves in discrete time steps. Each cell is characterized by an internal state whose value belongs to a finite set of possible packet trafficking and buffering. The updating of these states is

made at discrete times according to a local rule involving only a neighborhood of each cell. A lattice gas cellular automaton has specific neighbor interaction rules and is a model of a gas. Many complex physical systems such as traffic flow can be modeled as cellular automata. Cellular automata models can provide quantitative information for the complex systems being modeled.

A lattice gas model uses a set of simple “collision rules” to model the much more complex interactions of gas molecules. For the modeling of a network, the role of routers in the interaction of packet dynamics in the network suggests different collision rules.

We derive a model for the flow on a network accounting for the role of routers or similar control devices in determining the macroscopic features of the flow. We will proceed in analogy with lattice gas cellular automata by describing the interaction of particles (packets) arriving at nodes. The interactions in communication networks differ from those assumed for lattice gases, but we shall seek common features which lead to a fluid flow description in terms of spatial coordinates and time.

In order to set this work in perspective, we shall describe the lattice gas cellular automata model, including the continuum approximation and the derivation of the equation for the momentum flux (pressure) for the Euler equation. Then we discuss appropriate modifications to make the flow resemble flow on a communications network. We derive a lattice Boltzmann approach to a lattice gas residing on an irregular lattice, and discuss how this model would be modified to model network flows. This modeling effort will lead to equations for how packets move on a discrete set of nodes and links, moving at discrete times.

## 2.1 Lattice Gas-Cellular Automata Model

A lattice gas is a fluid model consisting of the motion of discrete particles on a regular lattice of nodes. A cellular automaton lattice gas model assumes that the gas has a finite number of particles and the particles move on links connecting nodes. At discrete time steps the particle moves through a link into the next node. If two or more particles meet at a node they collide. These collisions are governed by a set of rules. The set of rules take into account the spatial symmetry of the lattice and also conserve mass, momentum and energy.

## 2.2 Lattice Gas

In the lattice gas, the particle dynamics are governed by the equation,

$$n_i(x + c_i, t + 1) = n_i(x, t) + \Delta_i[n(x, t)] \quad (2.1)$$

where  $n_i$  represents the number of particles at lattice point  $x$  at time  $t$  with velocity  $c_i$  and  $\Delta_i[n(x, t)]$  are the collision operators. The collision operators are coded from a look-up table, and depend on the states of the nodes surrounding node  $x$ . For the model of a gas, the collision rules are designed to conserve mass, momentum and energy. One representative collision on a rectangular



two-dimensional lattice is shown in Figure 2.1. In this model, the four velocity directions ( $i = 1, 2, 3, 4$ ) are  $c_1 = i$ ,  $c_2 = j$ ,  $c_3 = -i$  and  $c_4 = -j$ . The collisions are encoded in the collision operator.

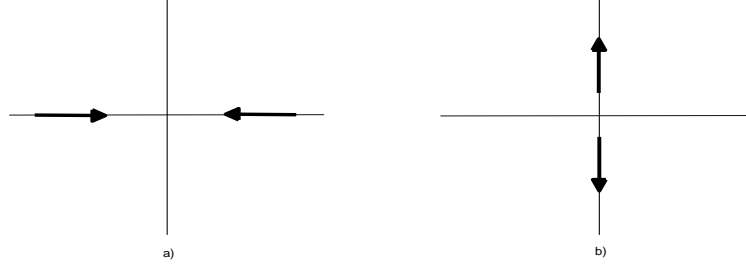


Figure 2.1: A typical collision on a rectangular lattice, a) before, and b) after.

We shall not consider energy here. Since the collisions conserve particles and momentum, we have

$$\begin{aligned}\sum_{i=1}^4 \Delta_i [n(x, t)] &= 0 \\ \sum_{i=1}^4 c_i \Delta_i [n(x, t)] &= 0 \\ \sum_{i=1}^4 \frac{1}{2} c_i^2 \Delta_i [n(x, t)] &= 0\end{aligned}$$

*Mass Equation*

Summing the particle dynamics equation (2.1) gives  $\sum_{i=1}^4 n_i(x + c_i, t + 1) = \sum_{i=1}^4 n_i(x, t)$ . Multiply the particle dynamic equation by  $c_i$  and sum that as well gives  $\sum_{i=1}^4 c_i n_i(x + c_i, t + 1) = \sum_{i=1}^4 c_i n_i(x, t)$ . Define the density as  $\rho(x, t) = \sum_{i=1}^4 n_i(x, t)$  and set the average velocity as

$$u(x, t) = \frac{\sum_{i=1}^4 c_i n_i(x, t)}{\rho(x, t)}$$

we can then sum the particle equation over all the nodes inside of some area  $V$  (see Figure 2.2), and obtain

$$\sum_{x \in V} \sum_{i=1}^4 [n_i(x, t+1) - n_i(x, t)] = - \left( \sum_{x \in V} \sum_{i=1}^4 [n_i(x + c_i, t+1) - n_i(x, t+1)] \right)$$

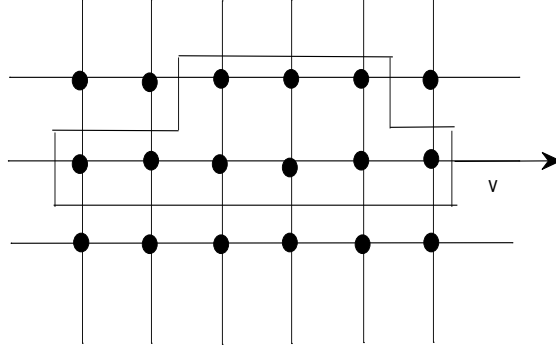


Figure 2.2: A “volume” in a rectangular lattice, with a normal vector shown on the right.

Note that the sum on the right hand side is the net particle flux out of the area (through the boundary). The sum on the right can be replaced by

$$\sum_{\substack{x \in V \\ x+c_i \notin V}} \sum_{i=1}^4 \nu \cdot c_i n_i(x, t+1)$$

where  $\nu$  is the unit normal to the boundary curve. Note that  $\nu \cdot c_i = -1, 0, 1$  depending on whether the velocity is directed into, along, or out of the boundary curve. The first term can be approximated by  $\sum_{x \in V} \frac{\partial \rho(x, t)}{\partial t}$ , and the flux term

can be approximated by  $\sum_{\substack{x \in V \\ x+c_i \notin V}} \nu \cdot \rho(x, t) u(x, t)$ . The spatial sums can be replaced by integrals, an area integral for the time derivative, and a line integral for the flux. We have

$$\int_V \frac{\partial \rho}{\partial t} dA = - \int_{\partial V} \nu \cdot \rho u ds$$

Using Gauss' theorem allows us to write this as

$$\int_V \left( \frac{\partial \rho}{\partial t} + \nabla \cdot \rho u \right) dA = 0$$

Assuming smoothness of the integrand, we have the equation of conservation of mass

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho u = 0$$

#### *Momentum Equation*

Multiply the particle equation (2.1) by  $c_i$  and sum gives

$$\sum_{x \in V} \sum_{i=1}^4 c_i [n_i(x, t+1) - n_i(x, t)] = - \sum_{\substack{x \in V \\ x+c_i \notin V}} \sum_{i=1}^4 (\nu \cdot c_i) c_i n_i(x, t+1) \quad (2.2)$$

Define the momentum flux tensor  $\Pi = - \sum_{i=1}^4 c_i c_i n_i(x, t)$  and following the argument above gives

$$\frac{\partial \rho u}{\partial t} = \nabla \cdot \Pi$$

The momentum flux can be written as  $\Pi = -\rho u u + \sigma$  where the stress  $\sigma$  can be expressed in microscopic terms as  $\sigma = - \sum_{i=1}^4 c_i' c_i' n_i(x, t)$ . Here  $c_i' = c_i - u$  is the velocity fluctuation. The momentum equation can be written as

$$\rho \left( \frac{\partial u}{\partial t} + u \cdot \nabla u \right) = \nabla \cdot \sigma$$

#### *Closure*

The behavior of the fluid is determined by specifying the stress  $\sigma$  in terms of the state variables, which are expressed in terms of  $\rho$  and  $u$ . A simple closure is to assume that the stress is isotropic, and given by  $\sigma_{ij} = -p \delta_{ij}$  where  $p = p(\rho, \theta)$  is the pressure, and  $\theta$  is the temperature. This approach requires the derivation of an energy equation, which is beyond the purpose of this discussion.

For lattice gases which are assumed to resemble a viscous fluid, closure is accomplished by assuming the distribution given by the particle dynamics equation is in steady state and homogeneous in space. The equation becomes

$\Delta_i \left[ n^{(ss)}(x, t) \right] = 0$ . For a viscous flow, the equilibrium distribution is assumed to depend on the shear rate. An appropriate dependence is assumed for small shear rate, and substituted into the particle dynamics equation. After assuming invariance, the coefficients are calculated, and substituted into the definition of stress. This approach is complicated by the fact that the expansion re-introduces terms expressing momentum flux by mass flux, i.e., a term proportional to  $uu$  [25].

### 3 A Probabilistic Model for Flow FIFO Buffer

In this Section, we describe a model for an individual buffer with unlimited memory operating on a FIFO (first in, first out) algorithm. In this model, packets are released based on their age  $a$ . We consider the probability  $P(a, k, \tau)$  of the buffer having  $k$  packets at age  $a$ , at time  $\tau$ . Packets at age  $a$  can exist in two states depending on whether  $a$  is the maximum age of packets in the buffer. We refer to the two states as either “buffered” or “send queue”. The buffered state refers to packets that are not being sent but remain in the buffer. The send queue state refers to packets at maximum age in the buffer, which are in the process of being sent. Figure 3.1 shows an example of packet behavior as they transition from the buffer to the send queue. Incoming bunches of packets are buffered at age zero, which is displayed by the figure on the right. At a certain age that is determined probabilistically, they transition to the send queue they begin to exit. Those packets that did not exit age in the buffer each time. The total probability of  $P(a, k, \tau)$  of  $k$  packets of age  $a$  in the buffer at time  $\tau$  is the probability of buffered  $k$  packets at age  $a$ ,  $B(a, k, \tau)$  in the queue and the probability of packets  $k$  in the send queue. Note that packets in the send queue at maximum age  $a = a_{max}$  in the process of being sent  $S(a, k, \tau)$ , so  $P(a, k, \tau) = S(a, k, \tau) + B(a, k, \tau)$ .  $P(a, k, \tau)$  is derived by understanding each probability  $B(a, k, \tau)$  and  $S(a, k, \tau)$ .

$B(a, k, t)$  is the probability of  $k$  packets buffered at age  $a$ . It is described by the following equation:

$$B(a, k, \tau + \Delta\tau) = B(a - 1, k, \tau)TB(a - 1, k, \tau) \quad (3.1)$$

This equation states that if  $k$  packets are buffered at age  $a - 1$  at time  $\tau$ , they will be buffered at age  $a$ , at time  $\tau + \Delta\tau$  with probability  $TB(a - 1, k, \tau)$ . Thus  $TB(a - 1, k, \tau)$  is the probability of transition of a buffered packet at age  $a - 1$  to age  $a$ , implying age  $a$  did not become the maximum age in the buffer.

$S(a, k, \tau)$  is the probability of  $k$  packets in the sending queue at  $a = a_{max}$  with the following equation describing it:

$$S(a, k, \tau + \Delta\tau) = S(a - 1, k + 1, \tau)TS(a - 1, k + 1, \tau) + B(a - 1, k, \tau)(1 - TB(a - 1, k, \tau)) \quad (3.2)$$

Equation (3.2) states that the probability of  $k$  packets going to the send queue is that either at time  $\tau$  they were already in the send queue but the number exceeded the link packet transport capacity, and therefore were not able to exit; or packets were previously in the buffered state and are now in the send queue.  $S(a, k, \tau)$  has two transition probabilities,  $TS(a - 1, k + 1, \tau)$  and  $1 - TB(a - 1, k, \tau)$ .  $TS(a - 1, k + 1, \tau)$  is the transition probability that if the number of packets  $k$  is greater then the link packet transport capacity, then packets that did not exit remain in the buffer and age, increasing  $a_{max}$  by 1. The probability  $TS(a - 1, k + 1, \tau)$  equals 1.

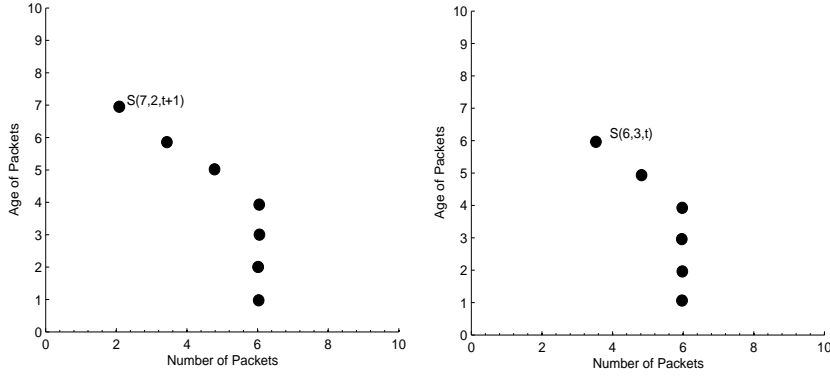


Figure 3.1: Example of Packets Being in the Send Queue

The other transition probability  $1 - TB(a - 1, k, \tau)$  corresponds to buffered packets entering the send queue. This quantity is the probability that at time  $\tau$ , the number of packets at maximum age  $a_{max}$  was below the link capacity (all packets were sent) and the maximum age dropped to a new lower value  $a_{max}(\tau + \Delta\tau) < a_{max}(\tau)$  where a new set of packets that were previously buffered will enter the send queue. In order for the maximum age to drop to a new lower value there could not have been packets between  $a_{max}(\tau)$  and  $a_{max}(\tau + \Delta\tau)$ . If  $B(a, k, \tau)$  is the probability of  $k$  buffered packets at age  $a$ , then  $1 - B(a, k, \tau)$  is the probability of there being no buffered packets at age  $a$ . The transition probability that the number of packets was below the link transport capacity and the maximum age dropped to a new lower value  $a$  is,

$$1 - TB(a - 1, k, \tau) = \sum_{a'=a+1}^{\infty} S(a', 1, \tau) \left(1 - \sum_{k^*=1}^{\infty} \sum_{a^*=a+1}^{a'-1} B(a^*, k^*, \tau)\right)$$

The transition probability  $TB(a-1, k, \tau)$  in equation (3.1) equals,

$$TB(a-1, k, \tau) = 1 - \sum_{a'=a+1}^{\infty} S(a', 1, t) \left(1 - \sum_{k*=1}^{\infty} \sum_{a*=a+1}^{a'-1} B(a^*, k^*, \tau)\right) \quad (3.3)$$

Substitute  $1 - TB(a-1, k, \tau)$  into equations (3.1) and (3.2), yielding

$$B(a, k, \tau + \Delta\tau) = B(a-1, k, \tau) \left(1 - \sum_{a'=a+1}^{\infty} S(a', 1, \tau) \left(1 - \sum_{k*=1}^{\infty} \sum_{a*=a+1}^{a'-1} B(a^*, k^*, \tau)\right)\right) \quad (3.4)$$

$$S(a, k, \tau + \Delta\tau) = S(a-1, k+1, \tau) + B(a-1, k, \tau) \sum_{a'=a+1}^{\infty} S(a', 1, \tau) \left(1 - \sum_{k*=1}^{\infty} \sum_{a*=a+1}^{a'-1} B(a^*, k^*, \tau)\right) \quad (3.5)$$

Equations (3.5) and (3.4) fully describes the two ways  $k$  packets arrive at age  $a$ . Figure 3.2 shows an example of the evolution of packets that are buffered in a queue. The figure on the left shows that packets in this cohort did not exit but rather aged in the buffer.

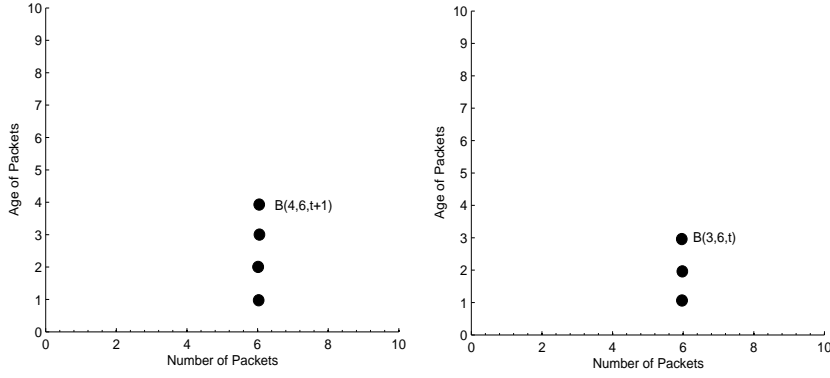


Figure 3.2: Example of Buffered Packets

Equations for  $S(a, k, \tau)$  and  $B(a, k, \tau)$  fully exhaust the possible states for packets, Buffered or Send Queue. These two equations describe packets at age  $a \geq 1$ . When packets arrive at a node they enter the system at age zero. They are either buffered or enter the send queue.  $S(0, k, \tau + \Delta\tau) + B(0, k, \tau + \Delta\tau)$  is the total probability of  $k$  packets at age zero. We assume that arrivals enter the Send Queue if the system is empty. The probability of being at maximum

age zero is the probability of finding the system empty,

$$\frac{S(0, k, \tau + \Delta\tau)}{S(0, k, \tau + \Delta\tau) + B(0, k, \tau + \Delta\tau)} = \sum_{a'=1}^{\infty} S(a', 1, \tau) \left( 1 - \sum_{k^*=1}^{\infty} \sum_{a^*=1}^{a'-1} B(a^*, k^*, \tau) \right) \quad (3.6)$$

If packets do not enter the send state but instead enter the buffered state at age zero then this probability is,

$$\frac{B(0, k, \tau + \Delta\tau)}{S(0, k, \tau + \Delta\tau) + B(0, k, \tau + \Delta\tau)} = 1 - \sum_{a'=1}^{\infty} S(a', 1, \tau) \left( 1 - \sum_{k^*=1}^{\infty} \sum_{a^*=1}^{a'-1} B(a^*, k^*, \tau) \right) \quad (3.7)$$

### 3.1 Steady State Analysis

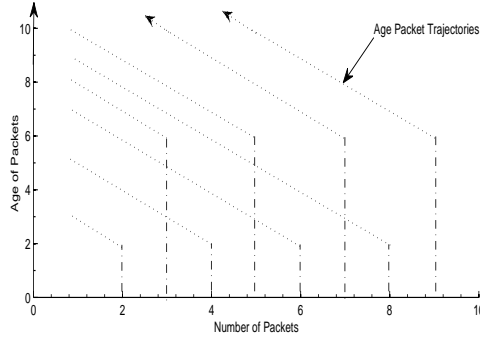


Figure 3.3: Age Packet Trajectories

We assume the probabilities are in steady state and the equations become, for age  $a \geq 1$

$$B(a, k) = B(a-1, k) \left( 1 - \sum_{a'=a+1}^{\infty} S(a', 1) \left( 1 - \sum_{k^*=1}^{\infty} \sum_{a^*=a+1}^{a'-1} B(a^*, k^*) \right) \right) \quad (3.8)$$

$$S(a, k) = S(a-1, k+1) + B(a-1, k) \sum_{a'=a+1}^{\infty} S(a', 1) \left( 1 - \sum_{k^*=1}^{\infty} \sum_{a^*=a+1}^{a'-1} B(a^*, k^*) \right) \quad (3.9)$$

and the equations at age zero become

$$\frac{S(0, k)}{S(0, k) + B(0, k)} = \sum_{a'=1}^{\infty} S(a', 1) \left( 1 - \sum_{k^*=1}^{\infty} \sum_{a^*=1}^{a'-1} B(a^*, k^*) \right) \quad (3.10)$$

$$\frac{B(0, k)}{S(0, k) + B(0, k)} = 1 - \sum_{a'=1}^{\infty} S(a', 1) \left( 1 - \sum_{k^*=1}^{\infty} \sum_{a^*=1}^{a'-1} B(a^*, k^*) \right) \quad (3.11)$$

### 3.2 Solution Method

An approach to finding the solution to the buffered and send probability equations is by using separation of variables for  $S(a, k)$  and  $B(a, k)$ ,

$$S(a, k) = N(a)U(k)$$

$$B(a, k) = A(a)T(k)$$

Set  $U(k+1) = \lambda U(k) = \lambda T(k)$ . Solving for  $U(k)$  we get  $U(k) = \frac{\alpha}{\lambda} \lambda^k$  where we set  $\frac{\alpha}{\lambda} = 1$ . This implies that  $T(k) = \lambda^k$ . Substituting into equations (3.8) and (3.9) yields

$$N(a) - \lambda N(a-1) = A(a-1) \left( \sum_{a'=a+1}^{\infty} N(a')U(1) \left( 1 - \sum_{k^*=1}^{\infty} \sum_{a^*=a+1}^{a'-1} A(a^*)T(k^*) \right) \right) \quad (3.12)$$

$$A(a) - A(a-1) = -A(a-1) \left( \sum_{a'=a+1}^{\infty} N(a')U(1) \left( 1 - \sum_{k^*=1}^{\infty} \sum_{a^*=a+1}^{a'-1} A(a^*)T(k^*) \right) \right) \quad (3.13)$$

The arrival parameter  $\lambda$  is determined by the traffic flow upstream the node being described, and is given by the network. Let  $U(1) = \lambda$  and the sum involving  $T(k)$  becomes by setting  $k = s + 1$ ,

$$\sum_{k=1}^{\infty} T(k) = \sum_{k=1}^{\infty} \lambda^k = \lambda \sum_{s=0}^{\infty} \lambda^s = \frac{\lambda}{1 - \lambda}$$

Equations (3.12) and (3.13) become

$$N(a) - \lambda N(a-1) = \lambda A(a-1) \left( \sum_{a'=a+1}^{\infty} N(a') \left( 1 - \frac{\lambda}{1 - \lambda} \sum_{a^*=a+1}^{a'-1} A(a^*) \right) \right) \quad (3.14)$$

$$A(a) - A(a-1) = -\lambda A(a-1) \left( \sum_{a'=a+1}^{\infty} N(a') \left( 1 - \frac{\lambda}{1 - \lambda} \sum_{a^*=a+1}^{a'-1} A(a^*) \right) \right) \quad (3.15)$$



We also apply separation of variables to equations (3.10) and (3.11), resulting in

$$\frac{N(0)}{N(0) + A(0)} = \sum_{a'=1}^{\infty} N(a') \left(1 - \frac{\lambda}{1-\lambda} \sum_{a^*=1}^{a'-1} A(a^*)\right) \quad (3.16)$$

$$\frac{A(0)}{N(0) + A(0)} = 1 - \sum_{a'=1}^{\infty} N(a') \left(1 - \frac{\lambda}{1-\lambda} \sum_{a^*=1}^{a'-1} A(a^*)\right) \quad (3.17)$$

The marginal probability of age  $a$  is

$$\begin{aligned} p(a) &= \sum_{k=0}^{\infty} (A(a)\lambda + N(a))\lambda^k \\ &= \frac{A(a)\lambda + N(a)}{1-\lambda} \end{aligned}$$

and the sum of all the marginal probabilities is equal to 1. This implies

$$\begin{aligned} \sum_{a=0}^{\infty} p(a) &= \sum_{a=0}^{\infty} \frac{A(a)\lambda + N(a)}{1-\lambda} \\ &= \sum_{a=0}^{\infty} \frac{\lambda}{1-\lambda} A(a) + \sum_{a=0}^{\infty} \frac{1}{1-\lambda} N(a) \\ &= 1 \end{aligned}$$

## 4 Discrete Network Model

A large communication network is composed of many hundreds to hundreds of thousands of routers distributing packets throughout the system. Graph theory describes the connectivity of the network [23, 24]. A graph is composed of a set  $N$  of nodes (vertices) and a set  $E$  of edges (links) where an edge connects a pair of nodes. In consequence,  $G = (N, E)$  is the underlying structure of a network. In a communication network, information travels between the nodes along the links that connect them. These links may be directed or undirected. In directed graphs a link has a preferred direction for information to travel, while in an undirected graph there is no distinction. The graph in this model is undirected; information travels in both directions. In this network model a matrix, called a route matrix, describes the path from source to destination for a packet to travel optimally in the network.

We simplify the model by assuming that routers are both host and switch computers, implying that packets are created, destroyed and buffered at the routers. Packets will flow through links to each router. Upon entering the network, a packet will go directly into their source buffer and will eventually exit the network at a pre-determined router. Each router instructs each resident packet which link to take as it travels to its destination.

## 4.1 Buffer State

The flow on this network is described by specifying  $b(j, d, a, \tau)$  the number of buffered packets at any time  $\tau$  at each node  $j \in N$ , with destination  $d \in N$ , of age  $a$  time units spent in the buffer.

## 4.2 Route Matrix

The network studied here is assumed to have packet routes determined statically. The route that a packet takes from router  $j$  to its destination  $d$  is determined based on predetermined knowledge of the network. We do not treat algorithms that change routes dynamically in response to traffic patterns. Specifically when a packet enters a network at a router, its complete path to its destination is determined.

The route matrix  $R_{j,d}$  is an  $N \times N$  matrix, where  $N$  is the total number of routers in the network. The entries  $R_{j,d}$  of the route matrix are the appropriate next router a packet will take as it propagates toward its destination. Specifically,  $R_{j,d} = x$  implies that for a packet at router  $j$  with address (destination)  $d$ ,  $x$  is the next router in the path [27].

## 4.3 Queue Dynamics

Each router contains a queue of buffered packets. Packets are generated and delivered at each router. When a packet is generated it is given a destination by its originator. Packets are assumed to have the same size. Packets enter a queue in one of two ways:

1. Flowing from another router along a connecting link.
2. Being generated at the router according to some given packet generator rate appropriate to that router.

Once a packet reaches its destination it is delivered, and exits immediately. We expect the number of packets in a queue to remain relatively low; otherwise, it is an indication of congestion. The number of packets entering the buffer at router  $j$  is described by the following equation,

$$b(j, d, 0, \tau + 1) = \sum_{a=0}^{\infty} \sum_{\substack{i=1 \\ j \neq d}}^N \delta_{j, R_{i,d}} \beta(i, d, a, \tau) + \nu(j, d, \tau) \quad (4.1)$$

Here  $\beta(i, d, a, \tau)$  denotes the number of packets sent from the node  $i$  toward destination  $d$  at time  $\tau$ . Each packet sent is chosen from the population of packets at the upstream nodes characterized by having age  $a$  in that buffer. The Kronecker delta connects the next location  $j$  from node  $i$  with destination  $d$  in terms of the route matrix. The second term on the right side,  $\nu(j, d, \tau)$ , describes the number of new packets entering the system at node  $j$ , with destination  $d$ , at time  $\tau$ . Packets entering the buffer have age  $a = 0$ .

We assume that each router uses a FIFO unlimited memory buffer where packets are released based on their age  $a$ . The number of packets in the buffer are described by the evolution equation

$$b(j, d, a + 1, \tau + 1) = b(j, d, a, \tau) - \beta(j, d, a, \tau) \quad (4.2)$$

The second term on the right hand side  $\beta(j, d, a, \tau)$ , determines the number of packets that are sent outward from node  $j$  with destination  $d$  at each time.

At any router, destinations are competing for the same link to a router. This creates competition among destinations as to the number of packets that will have access to the link. Due to this competition an algorithm was created to describe the flow of packets to a link. We note that if a packet buffered at  $j$  with destination  $d$  shares the next link  $R_{j,d}$  with a packet buffered at  $j$  with destination  $l$ , there is a possible conflict with sending both packets.

#### 4.4 The Flow Variable

To determine the number of packets that flow  $\beta(j, d, a, \tau)$  from node  $j$ , with destination  $d$ , we have to determine the number of packets that are competing at each time for a link. We define a function  $k(j, d, a, \tau)$ , which sums the total number of packets having destinations which compete for the same next link at age  $a$ . The total number of packets  $k(j, d, a, \tau)$ , to be sent from  $j$  to  $R_{j,d}$  at age  $a$  is,

$$k(j, d, a, \tau) = \sum_{l=1}^N b(j, l, a, \tau) \delta_{R_{j,l}, R_{j,d}} \quad (4.3)$$

The Kronecker delta in equation (4.3) separates the links that are competing with destination  $d$  by using the route matrix. The buffer at a router needs to know which destinations are competing for the same link to the next router, since each link has a packet capacity. If destinations are competing and there are packets ready to exit, we use Kronecker delta function to distinguish which destinations are competing for the same next router.

Mathematically, the competition for two destinations  $d$  and  $l$  is given by

$$\delta_{R_{j,l}, R_{j,d}} = \begin{cases} 1, & \text{if } R_{j,d} = R_{j,l}; \\ 0, & \text{if } R_{j,d} \neq R_{j,l}. \end{cases}$$

The number of packets flowing from node  $j$  to node  $R_{j,d}$  is modeled as,

$$\beta(j, d, a, \tau) = b(j, d, a, \tau) c(j, d, a, \tau) \quad (4.4)$$

$\beta(j, d, a, \tau)$  is the number of packets at age  $a$  with destination  $d$  that will be flowing away from node  $j$ . The competition function  $c(j, d, a, \tau)$  determines the fraction of packets at age  $a$  that will be sent. The competition function equals

$$c(j, d, a, \tau) = \begin{cases} b_0, & \text{if } a > a^*; \\ \frac{b_0 - K(j, d, a, \tau)}{k(j, d, a, \tau)}, & \text{if } a = a^*; \\ 0, & \text{if } a < a^*. \end{cases}$$

where  $b_0$  is the bandwidth or total link packet transport capacity. The function  $k(j, d, a, \tau)$  determines the total number of packets at each age  $a$  competing for the same next link,  $a^*$  is the minimum age in the buffer that can send packets, and

$$K(j, d, a, \tau) = \sum_{\acute{a}=a+1}^{\infty} k(j, d, \acute{a}, \tau)$$

is the number of packets above age  $a$ . Consequently, if for some  $a$ ,  $K(j, d, a, \tau) \leq b_0$  and  $K(j, d, a-1, \tau) > b_0$ , then we define  $a^* = a$ . All packets at ages greater than  $a^*$  are sent, hence  $c(j, d, a, \tau) = 1$  for  $a > a^*$ . However, if the number of packets being sent at ages above  $a^*$  is below  $b_0$  then some of the packets at  $a^*$  also contribute to the total number of packets that can be sent up to the capacity  $b_0$ . Then  $b_0 - K(j, d, a^*, \tau)$  is the excess number of packets that can be sent at age  $a^*$ , and since destinations are competing for the same next link,  $c(j, d, a^*, \tau)$  determines the fraction of packets that can be sent at each destination. Packets with age less than  $a^*$  will not be sent since the total number of packets being sent has reached link capacity and consequently  $\beta(j, d, a, \tau) = 0$  for  $a < a^*$ . If  $a^* = 0$ , then  $K(j, d, 0, \tau) \leq b_0$  resulting in all packets exiting.

#### 4.5 Discrete Flow Equation

The number of packets at router  $j$  evolves in time as packets are flowing throughout a network. Let  $n(j, d, \tau)$  be the number of buffered messages at router  $j$  with destination  $d$  at time  $\tau$ . Then

$$n(j, d, \tau) = \sum_{a=0}^{\infty} b(j, d, a, \tau) \quad (4.5)$$

so that the number of packets at router  $j$  changes in time according to

$$n(j, d, \tau + 1) - n(j, d, \tau) = \sum_{a=0}^N b(j, d, a, \tau + 1) - \sum_{a=0}^N b(j, d, a, \tau) \quad (4.6)$$

Using the evolution equations (4.2), equation (4.6) can be rewritten and expanded as

$$\begin{aligned} n(j, d, \tau + 1) - n(j, d, \tau) &= \sum_{a=0}^{\infty} (b(j, d, a, \tau) - \beta(j, d, a, \tau)) \\ &\quad + b(j, d, 0, \tau + 1) - \sum_{a=0}^{\infty} b(j, d, a, \tau) \end{aligned} \quad (4.7)$$

Also substituting equation (4.1) for  $b(j, d, 0, \tau + 1)$  into equation (4.7), we have

$$\begin{aligned}
n(j, d, \tau + 1) - n(j, d, \tau) = & - \sum_{a=0}^{\infty} \beta(j, d, a, \tau) + \sum_{a=0}^{\infty} \sum_{\substack{i=1 \\ j \neq d}}^N \delta_{j, R_{i,d}} \beta(i, d, a, \tau) \\
& + \nu(j, d, \tau) - \sum_{a=0}^{\infty} \beta(j, j, a, \tau)
\end{aligned} \tag{4.8}$$

Equation (4.8) is the discrete equation expressing “conservation” of packets at node  $j$ . The first term on the right side of the equal sign is the number of packets that flowed away from node  $j$ . The second term describes the number of new packets into node  $j$  from the surrounding connected nodes. Packets whose destination is router  $j$  exit immediately. This is expressed by the term,  $-\beta(j, j, a, \tau)$  which becomes the sink term at time  $\tau$ . The third term is the source.

## 5 Continuum Description

### 5.1 Geometry

The spatial location of each router in a computer network is of importance in some applications. A router serves a particular coverage area, distributing packets to and from users who are using the network. To describe the spatial and temporal behavior of a network, routers are associated with physical spatial locations. Let  $\mathbf{x}_j$  be the location of router  $j$ . Let  $\mathbf{y}$  be the location of destination  $d$ . Each  $\mathbf{x}_j$  generates a Voronoi polygon  $V(\mathbf{x}_j)$ , a tessellation of the plane [16]. The Voronoi polygon  $V(\mathbf{x}_j)$  is the set of points closer to the point (router)  $\mathbf{x}_j$ , than to all the other routers with location  $\mathbf{x}_k, k \neq j$  in the domain.

Each router  $\mathbf{x}_j$  shares physical boundaries with another set of routers  $\mathbf{x}_b(\mathbf{x}_j)$  so that packets moving through the network will pass through physical boundaries around routers. Points on the boundary of the Voronoi polygon are equal distance from the two routers that define the boundary.

Given  $P$ , a finite set of distinct points in the Euclidean plane, each point generates a Voronoi polygon in the plane. A collection of such polygons in Euclidean space define a Voronoi Diagram  $V_P$ . The set given by

$$V_P = \bigcup_{j \in P} V(\mathbf{x}_j)$$

is the planar ordinary Voronoi diagram generated by  $P$  (or the Voronoi diagram of  $P$ ). We denote the area of the Voronoi polygon by  $|V(\mathbf{x}_j)|$ .

When a network is created a route matrix is determined for optimal paths. The route matrix indicates which routers are communicating directly with each other. Communicating routers do not necessarily share a boundary. Boundaries are defined by the Voronoi polygons generated by a router and its spatial

neighbors. We are interested in the rate of change of the number of packets in a Voronoi diagram  $V_P$  and the flow of packets entering and exiting the boundaries. We are not, however, interested in the flow internal to the Voronoi diagram between routers and their Voronoi polygons  $V(\mathbf{x}_j), j \in P$ .

In order to describe the flow into and out of a spatial region, consider a Voronoi diagram  $V_P$  and its boundary  $\partial V_P$ . The boundary is made up of several straight line segments, each of which is a shared boundary between the two Voronoi polygons,  $V(\mathbf{x}_l)$  and  $V(\mathbf{x}_m)$  with  $l \in P$  and  $m \notin P$ . We denote the boundary element by  $\partial V_{l,m}$ , and its length by  $|\partial V_{l,m}|$ . Then  $\partial V_P = \bigcup_{\substack{l \in P \\ m \notin P}} \partial V_{l,m}$

## 5.2 Density

Routers are endowed with physical spatial locations in order to derive flow equations. Let  $V_P$  be a Voronoi diagram and  $\partial V_P$  its boundary. If  $n(j, d, \tau)$  is the number of buffered packets at node  $j$  (location  $\mathbf{x}_j$ ) with destination  $d$  (location  $\mathbf{y}$ ), then we define  $\rho(\mathbf{x}_j, \mathbf{y}, t)$  as the density of packets associated with Voronoi polygon  $V(\mathbf{x}_j)$  by

$$\rho(\mathbf{x}_j, \mathbf{y}, t) = \frac{n(j, d, \tau)}{|V(\mathbf{x}_j)|} \quad (5.1)$$

where  $|V(\mathbf{x}_j)|$  denotes the area of  $V(\mathbf{x}_j)$  and  $t = \tau \Delta t$  and  $0 \leq t \leq \infty$ .

We assume that  $\rho(\mathbf{x}_j, \mathbf{y}, t)$  is a smooth function of the spatial variables  $\mathbf{x}_j$  and  $\mathbf{y}$  and time  $t$ . Consider the number of packets buffered in the Voronoi diagram  $V_P$ :

$$\sum_{j \in P} n(j, d, \tau)$$

which then equals

$$\sum_{j \in P} n(j, d, \tau) = \sum_{j \in P} \rho(\mathbf{x}_j, \mathbf{y}, t) |V(\mathbf{x}_j)|$$

The continuum limit of this quantity is

$$\sum_{j \in P} n(j, d, \tau) \rightarrow \int_{V_P} \rho(\mathbf{x}, \mathbf{y}, t) dV \quad (5.2)$$

where  $\rightarrow$  denotes an appropriate continuum limit.

## 5.3 Flow Equation

We derive an evolution equation for  $\rho(\mathbf{x}_j, \mathbf{y}, t)$  by summing equation (4.8) over  $j \in P$  and dividing by  $\Delta t$ . Thus,

$$\int_{V_P} \frac{\rho(\mathbf{x}, \mathbf{y}, t + \Delta t) - \rho(\mathbf{x}, \mathbf{y}, t)}{\Delta t} dV = \sum_{j \in P} \frac{n(j, d, \tau + 1) - n(j, d, \tau)}{\Delta t} \quad (5.3)$$

where

$$\begin{aligned} \sum_{j \in P} \frac{n(j, d, \tau + 1) - n(j, d, \tau)}{\Delta t} &= \sum_{j \in P} \sum_{a=0}^{\infty} \frac{1}{\Delta t} (-\beta(j, d, a, \tau) + \\ &\quad \sum_{\substack{i \in P \\ i \neq j}}^N \delta_{j, R_{i, d}} \beta(i, d, a, \tau) \\ &\quad + \nu(j, d, \tau) - \beta(j, j, a, \tau)) \end{aligned} \quad (5.4)$$

If we take the limit as  $\Delta t$  goes to zero, the first term in equation (5.4) becomes

$$\int_{V_P} \frac{\partial \rho}{\partial t} dV$$

## 5.4 Flux

We consider the flux terms in the evolution equation. We focus on one Voronoi diagram boundary element  $\partial V = \partial V_{l, m}$ , where  $l \in P$  and  $m \notin P$  with  $l$  and  $m$  share a boundary of their Voronoi polygons. We define the flow direction between routers in terms of the unit vectors connecting router  $\mathbf{x}_i$  to router  $\mathbf{x}_j$  as  $\frac{\mathbf{x}_i - \mathbf{x}_j}{|\mathbf{x}_i - \mathbf{x}_j|}$ . The flow vector from node  $\mathbf{x}_j$  to node  $\mathbf{x}_i$  is then

$$\frac{\mathbf{x}_i - \mathbf{x}_j}{|\mathbf{x}_i - \mathbf{x}_j|} \beta(j, d, a, \tau)$$

Note that  $i = R_{j, d}$ , and define

$$\delta_{(j, i)|(l, m)} = \begin{cases} 1, & \text{if the line segment connecting } \mathbf{x}_j \text{ to } \mathbf{x}_i \text{ passes through } \partial V_{l, m}; \\ 0, & \text{otherwise.} \end{cases}$$

This counter determines whether the flux from router  $\mathbf{x}_j$  to router  $\mathbf{x}_i$  passes through  $\partial V_{l, m}$ . Define  $\mathbf{n}_{l, m}$  as the unit normal vector on boundary  $\partial V_{l, m}$  pointing in the direction of the adjacent router  $\mathbf{x}_m$  outside of  $V$  but communicating with  $\mathbf{x}_l$ . Then  $\mathbf{n}_{l, m} = \frac{\mathbf{x}_m - \mathbf{x}_l}{|\mathbf{x}_m - \mathbf{x}_l|}$  is the unit exterior normal to boundary element.

In equation (5.4) the outgoing flux term becomes

$$\sum_{a=0}^{\infty} \sum_{\substack{j \in P \\ i \notin P}} \frac{-1}{\Delta t} \frac{\mathbf{x}_i - \mathbf{x}_j}{|\mathbf{x}_i - \mathbf{x}_j|} \beta(j, d, a, \tau)$$

We can define the outflow per unit area  $\Phi$  through boundary element  $\partial V_{l, m}$  with length  $|\partial V_{l, m}|$  as

$$\sum_{a=0}^{\infty} \sum_{\substack{j \in P \\ i \notin P}} \delta_{(j, i)|(l, m)} \delta_{i, R_{j, d}} \frac{-1}{\Delta t} \frac{\mathbf{x}_i - \mathbf{x}_j}{|\mathbf{x}_i - \mathbf{x}_j|} \beta(j, d, a, \tau) \cdot \mathbf{n}_{l, m} = \Phi^O(\mathbf{x}_{l, m}, \mathbf{y}, t) \cdot \mathbf{n}_{l, m} |\partial V_{l, m}| \quad (5.5)$$

where  $\mathbf{x}_{l,m}$  is the midpoint of  $\partial V_{l,m}$ , the unit vector normal  $\mathbf{n}_{l,m} = \frac{\mathbf{x}_i - \mathbf{x}_j}{|\mathbf{x}_i - \mathbf{x}_j|}$  and  $\Phi^O(\mathbf{x}_{l,m}, \mathbf{y}, t)$  is outgoing flux. Again, we assume that  $\Phi^O(\mathbf{x}_{l,m}, \mathbf{y}, t)$  is a smooth function of  $\mathbf{x}$ ,  $\mathbf{y}$  and  $t$ .

A similar equation for the incoming flux term is given by

$$\sum_{a=0}^{\infty} \sum_{\substack{i \notin P \\ j \in P}} \delta_{j,R_{i,d}} \delta_{(j,i)|(l,m)} \frac{1}{\Delta t} \frac{\mathbf{x}_i - \mathbf{x}_j}{|\mathbf{x}_i - \mathbf{x}_j|} \beta(i, d, a, \tau) \cdot \mathbf{n}_{l,m} = \Phi^I(\mathbf{x}_{l,m}, \mathbf{y}, t) \cdot \mathbf{n}_{l,m} \mid \partial V_{l,m} \mid \quad (5.6)$$

The flux of packets entering and exiting the boundary  $\partial V_{l,m}$  is

$$\Phi(\mathbf{x}_{l,m}, \mathbf{y}, t) \cdot \mathbf{n}_{l,m} \mid \partial V_{l,m} \mid = \Phi^O(\mathbf{x}_{l,m}, \mathbf{y}, t) \cdot \mathbf{n}_{l,m} \mid \partial V_{l,m} \mid - \Phi^I(\mathbf{x}_{l,m}, \mathbf{y}, t) \cdot \mathbf{n}_{l,m} \mid \partial V_{l,m} \mid$$

with  $\Phi^I(\mathbf{x}_{l,m}, \mathbf{y}, t)$  as the incoming flux terms. When we sum over all  $l \in P$  and  $m \notin P$ , we get the total flux of packets entering and exiting a Voronoi diagram  $V_P$  with destination  $\mathbf{y}$

$$\sum_{\substack{l \in \partial P \\ m \in \partial P_l}} \Phi(\mathbf{x}_{l,m}, \mathbf{y}, t) \cdot \mathbf{n}_{l,m} \mid \partial V_{l,m} \mid \quad (5.7)$$

where  $\partial P = \{l \in P \mid V(\mathbf{x}_l) \cap V(\mathbf{x}_m) \not\subseteq \emptyset \mid m \notin P\}$  and  $\partial P_l = \{m \notin P \mid V(\mathbf{x}_l) \cap V(\mathbf{x}_m) \not\subseteq \emptyset\}$ .

This sum gives the net flux across the boundary of a Voronoi Diagram  $V_P$  into  $V_P$ , which becomes an integral in the continuum limit

$$\sum_{\substack{l \in \partial P \\ m \in \partial P_l}} \Phi(\mathbf{x}_{l,m}, \mathbf{y}, t) \cdot \mathbf{n}_{l,m} \mid \partial V_{l,m} \mid \rightarrow \int_{\partial V_P} \Phi(\mathbf{x}, \mathbf{y}, t) \cdot \mathbf{n} ds$$

where  $\mid \partial V_{l,m} \mid$  becomes  $ds$ .

## 5.5 Source and Sink

The derivation for the source and sink terms are very similar. We can sum the  $\nu(j, d, \tau)$  terms, the source term for new packets entering the Voronoi Diagram  $V_P$ , with destination  $\mathbf{y}$ , from equation (4.8)

$$\sum_{j \in P} \frac{1}{\Delta t} \frac{\nu(j, d, \tau)}{|V(\mathbf{x}_j)|} \mid V(\mathbf{x}_j) \mid \quad (5.8)$$

We can do the same thing for the sink terms

$$\sum_{a=0}^{\infty} \sum_{j \in P} \frac{1}{\Delta t} \frac{\beta(j, j, a, \tau)}{|V(\mathbf{x}_j)|} \mid V(\mathbf{x}_j) \mid \quad (5.9)$$

If we define the continuum limit of

$$\frac{1}{\Delta t} \frac{\beta(j, j, a, \tau)}{|V(\mathbf{x}_j)|} = \gamma(\mathbf{x}, \mathbf{y}, t) \quad (5.10)$$



we have

$$\sum_{j \in P} \frac{1}{\Delta t} \nu(j, d, \tau) \rightarrow \int_{V_P} \gamma(\mathbf{x}, \mathbf{y}, t) dV \quad (5.11)$$

We can do the same to the sink term, resulting in

$$\sum_{a=0}^{\infty} \sum_{j \in P} \frac{1}{\Delta t} \beta(j, j, a, \tau) \rightarrow \int_{V_P} \sigma(\mathbf{x}, t) dV \quad (5.12)$$

## 5.6 Continuity Equation

Combining the integral of the density changing in time, the flux term (5.7) and the source and sink terms (5.11) and (5.12) we arrive at the equation for conservation of packets for Voronoi Diagram  $V_P$ :

$$\frac{d}{dt} \int_{V_P} \rho(\mathbf{x}, \mathbf{y}, t) dV = - \int_{\partial V_P} \Phi(\mathbf{x}, \mathbf{y}, t) \cdot \mathbf{n} ds + \int_{V_P} \gamma(\mathbf{x}, \mathbf{y}, t) dV - \int_{V_P} \sigma(\mathbf{x}, t) dV \quad (5.13)$$

Assuming sufficient smoothness and using Leibnitz rule and the divergence theorem gives

$$\int_V \left( \frac{\partial \rho}{\partial t} + \nabla \cdot \Phi(\mathbf{x}, \mathbf{y}, t) - \gamma(\mathbf{x}, \mathbf{y}, t) + \sigma(\mathbf{x}, t) \right) dV = 0 \quad (5.14)$$

Again, for sufficient smoothness of the density, flux, source density and sink density, we can use the Dubois Reymond- Lemma to derive the continuity equation,

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \Phi(\mathbf{x}, \mathbf{y}, t) = \gamma(\mathbf{x}, \mathbf{y}, t) - \sigma(\mathbf{x}, t) \quad (5.15)$$

## 5.7 Connection: Transport Model and Cellular Automata Lattice Gas Model

When we consider a network of nodes connected by links, the spatial location of the nodes is not regular, and the links are not restricted to lie in regular directions. In analogy with the cellular automata model for a gas, we shall assume that packets are of uniform size, and are not buffered but sent out immediately. In this case,  $b(j, d, 0, \tau) = n(j, d, \tau)$  and  $b(j, d, a, \tau) = 0$  for  $a > 0$ . Moreover, the velocity of the particles (packets) is determined by the physical locations and the propagation speed of the links.

In analogy with the particle dynamic equation (2.1), we apply the above assumptions to equation (4.8) without sources or sinks. We also note that in this case, the age variable  $a$  is always equal to zero, since no packets are buffered,

and the flow variable is

$$\begin{aligned}
n(j, d, \tau + 1) - n(j, d, \tau) &= - \sum_{a=0}^{\infty} \beta(j, d, a, \tau) + \sum_{a=0}^{\infty} \sum_{\substack{i=1 \\ j \neq d}}^N \delta_{j, R_{i,d}} \beta(i, d, a, \tau) \\
&= -b_0 b(j, d, 0, \tau) + \sum_{\substack{i=1 \\ j \neq d}}^N \delta_{j, R_{i,d}} b_0 b(i, d, 0, \tau) \\
&= -b_0 n(j, d, \tau) + \sum_{\substack{i=1 \\ j \neq d}}^N \delta_{j, R_{i,d}} b_0 n(i, d, \tau)
\end{aligned}$$

Figure (5.1) shows the “collision” model. Continuing the analogy with the lattice gas, we define the velocity along each link as  $\mathbf{c}_{i,j} = b_0 \frac{\mathbf{x}_i - \mathbf{x}_j}{|\mathbf{x}_i - \mathbf{x}_j|}$ , and equation (5.16) can be written as

$$n(j, d, \tau + 1) - n(j, d, \tau) - \mathbf{c}_{j, \mathbf{R}_{j,d}} \mathbf{n}(\mathbf{j}, \mathbf{d}, \tau) \cdot \boldsymbol{\nu} + \sum_{\substack{i=1 \\ j \neq d}}^N \delta_{j, \mathbf{R}_{i,d}} \mathbf{c}_{i,j} \mathbf{n}(\mathbf{i}, \mathbf{d}, \tau)$$

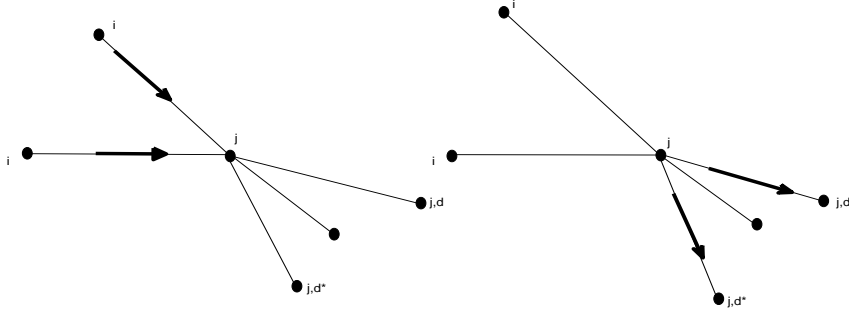


Figure 5.1: Typical interaction on a non-uniform lattice.

## 6 One-Dimensional Flow Model

Let us consider the one-dimensional continuity equation flow between a given source, taken as  $x = 0$  and with a single destination at  $x = y$ . The equation is then

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x} \Phi(x, y, t) = 0 \tag{6.1}$$

for  $0 < x < y$ , with  $\rho(x, y, t) \equiv 0$  otherwise.

If we assume that  $y > x$ , the flow goes from left to right and the left and right boundary points are the source and sink respectively. We are interested how the density changes at the nodes between the source and the sink. The equations are non-dimensionalized so that in the non-saturated one dimensional model the flow variable  $\beta(j, a, \tau)$  is equal to the density  $\rho(x, t)$  and the maximum saturation flow is equal to one.

We shall assume that the flux in one dimension in this model is given by

$$\Phi(x, y, t) = \begin{cases} \rho(x, y, t) & \text{if } \rho(x, y, t) < \Phi_{max}(x, y, t) \\ \Phi_{max}(x, y, t) & \text{if } \rho(x, y, t) \geq \Phi_{max}(x, y, t) \end{cases}$$

The flow saturation is  $\Phi_{max}(x, y, t)$  is assumed to be  $\leq 1$ . We refer to  $\Phi_{max}(x, y, t) = 1$  as system saturation. We wish to study situations where  $\Phi_{max}(x, y, t) < 1$  for limited regions and/or time intervals, representing

We will use the method of characteristics to solve the continuity equation. When the flow is non-saturated, then  $\Phi(x, t) = \rho(x, t)$ . Our one-dimensional continuity equation (20) becomes

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho}{\partial x} = 0 \quad (6.2)$$

When we solve equation (21) using the method of characteristics and with initial condition  $\rho_0(x, 0)$ , our solution to equation (21) is  $\rho(x, t) = \rho_0(x - t)$ . This solution is valid unless there is a discontinuity in the density or the flux. We explore several cases below.

## 6.1 Non Saturated Flow

In this example we begin with an initial density of packets, and as time progresses no new packets are introduced into the flow. We will examine how the flow is progressing and watch how the density at each spatial grid point changes. Since the flux density  $\Phi(x, t) = \rho(x, t)$ , no packets should stay at a spatial grid point since they are all below the link packet transport capacity  $\Phi_{max}(x, t)$ . The function  $\rho_0$  is given by

$$\rho_0(x) = \begin{cases} 0.2 & 40 \leq x \leq 80 \\ 0.4 & 0 \leq x < 40 \\ 0 & x < 0 \end{cases}$$

Figure 6.1 displays the initial condition and Figures 6.2, 6.3 and 6.4 show the flow progressing in time as a wave moving from left to right.

For this simple flow, the model performed as expected.

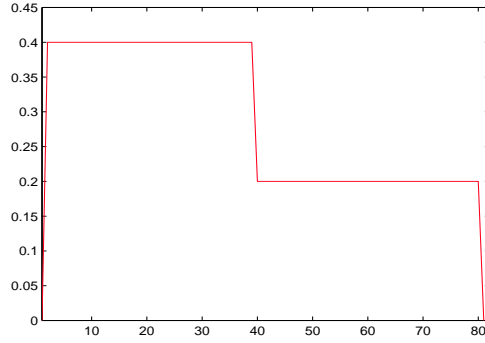


Figure 6.1: Packet Density , Initial Condition

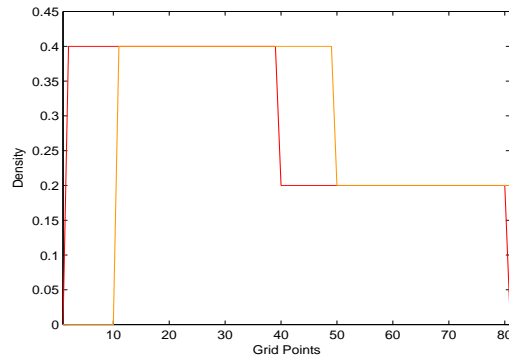


Figure 6.2: Packet Density, Time: 0,10,20

## 6.2 Flow with Saturation

The flow model in the previous section dealt with a flow where  $\rho(x, y, t)$  was less than the link packet transport capacity. Using the same initial condition as the previous section we will examine an example where  $\rho(x, y, t)$  can be greater than the link packet transport capacity. We assume that the source point  $x = 0$  is emitting packets into the system as follows

$$\nu(0, 80, t) = \begin{cases} 1.25 & t \leq 30 \\ 0.5 & t > 30 \end{cases}$$

We now have the solution

$$\rho(x, t) = \delta(x)f(t) + \rho_0(x - t)$$

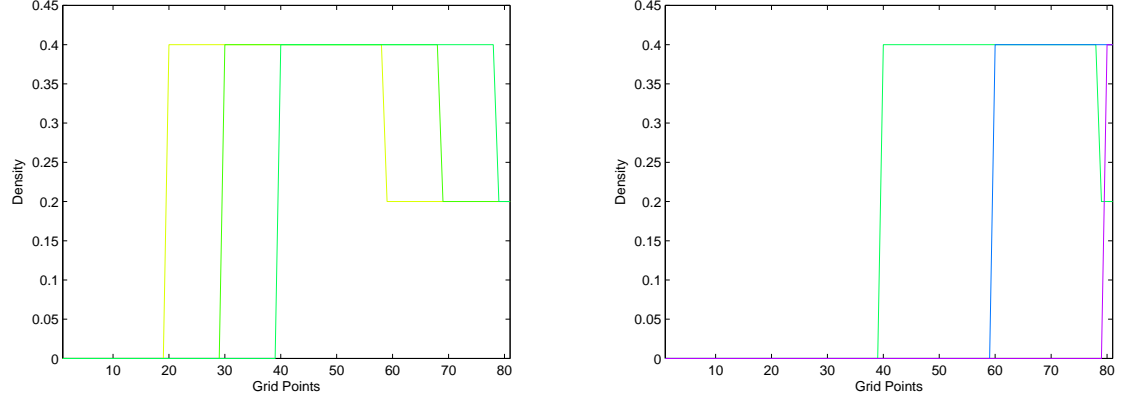


Figure 6.3: Packet Density Scenario, Time: 20,30,40, Time: 40,60,80

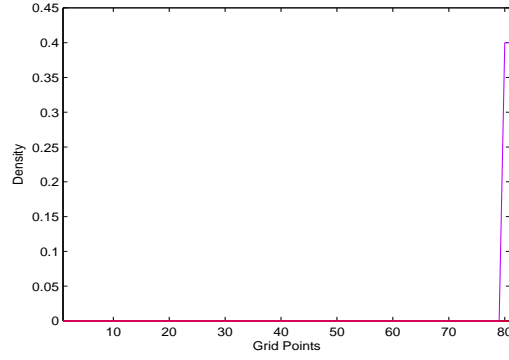


Figure 6.4: Packet Density, Time: 80,90,100

When  $\delta(x)$ , with  $f(t)$  equal to

$$f(t) = \begin{cases} (1.25 - 1.00)t & 0 < t \leq 30 \\ 30(1.25 - 1.00) - (t - 30)0.5 & 30 < t \leq 45 \\ 0 & t > 45 \end{cases}$$

and the function  $\rho_0(x)$  equals

$$\rho_0(x) = \begin{cases} 0.2 & 40 \leq x \leq 80 \\ 0.4 & 0 < x < 40 \\ 1 & -45 \leq x < 0 \\ 0.5 & -\infty < x < -45 \end{cases}$$

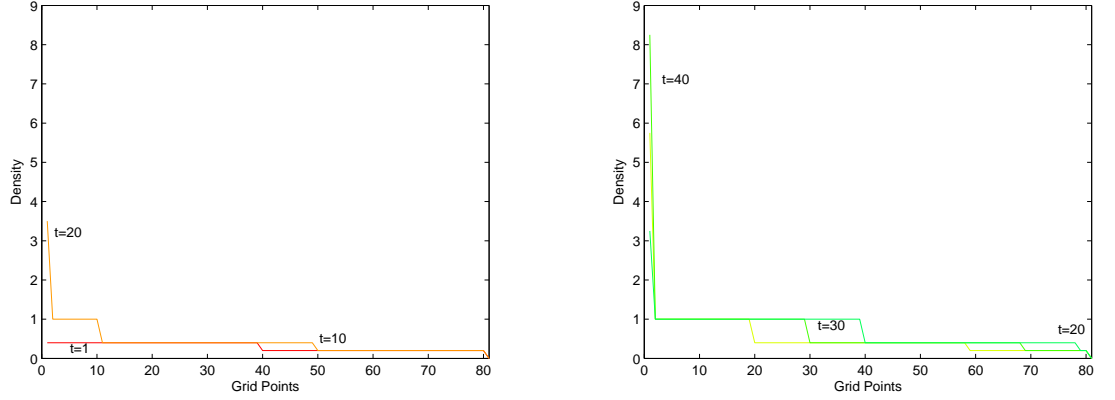


Figure 6.5: Packet Density Scenario, Time: 0,10,20, Time: 20,30,40

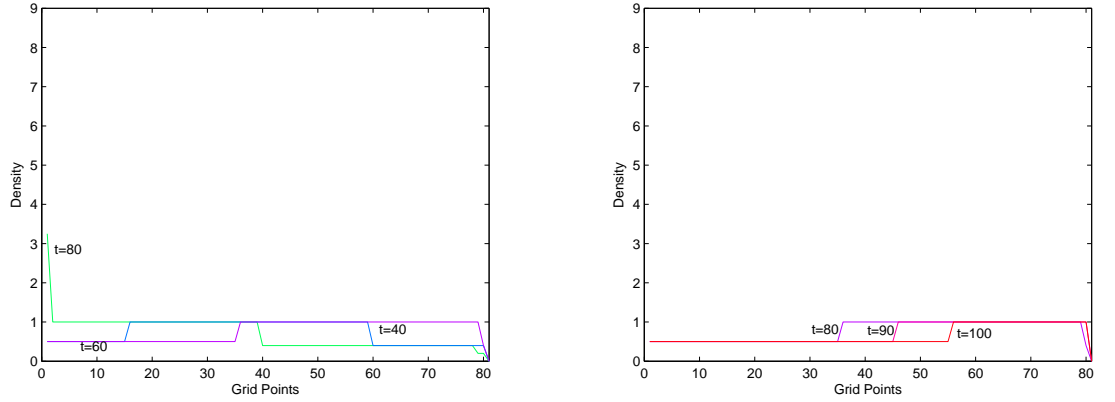


Figure 6.6: Packet Density Scenario, Time: 40,60,80, Time: 80,90,100

Examining the flow with saturation we notice that these are packets buffered at  $x=0$ , and that the flow remains at the link packet transport capacity for some time for an interval to the right of  $x = 0$ , but the buffered packets clear the node at  $x = 0$  after, and the flow eventually remains at 0.5.

### 6.3 Interruptions: Disturbances in the Flow

One of the issues we wanted to understand in this network model is what happens to the network flow when there is a disruption of some sort in the flow. In this example we examine interruptions to the flow in one dimensional models. Interruptions in this model might occur in one of two types, (i) limited

bandwidth or more specifically, the link packet transport capacity drops to a lower value; or (ii) router interruptions, when the routers corresponding to a grid point are down for some time. We use the same initial condition as before to investigate interruptions. This model scheme may have several interruptions that can take place.

The source node has a constant source resulting in a density downstream corresponding to 0.4 packets. Down town is experienced around the point  $x = 15$ . This down time equals  $\Phi_{max}(x, t)$  equals

$$\Phi_{max}(15, t) = \begin{cases} 0 & 1 \leq t \leq 20 \\ 1 & \text{otherwise} \end{cases}$$

The solution is given by  $\rho(x, t) = \delta(x - 15)f(t) + \rho_0(x - t)$  where  $f(t)$  is given by

$$f(t) = \begin{cases} 0.4t & t \leq 20 \\ 0.4(20) - 0.6(t - 20) & 20 < t < 30 \\ 0 & t > 30 \end{cases}$$

and  $\rho_0(x)$  is

$$\rho_0(x) = \begin{cases} 0.2 & 40 \leq x \leq 80 \\ 0.4 & 0 < x < 40 \\ 1 & -33 \leq x < 0 \\ 0.4 & -\infty < x < -33 \end{cases}$$

The figures are side by side in this example where one shows the density in “normal” resolution, while the other shows a close up of the neighborhood of the grid points where  $\Phi_{max}(x, t) = 1$ .

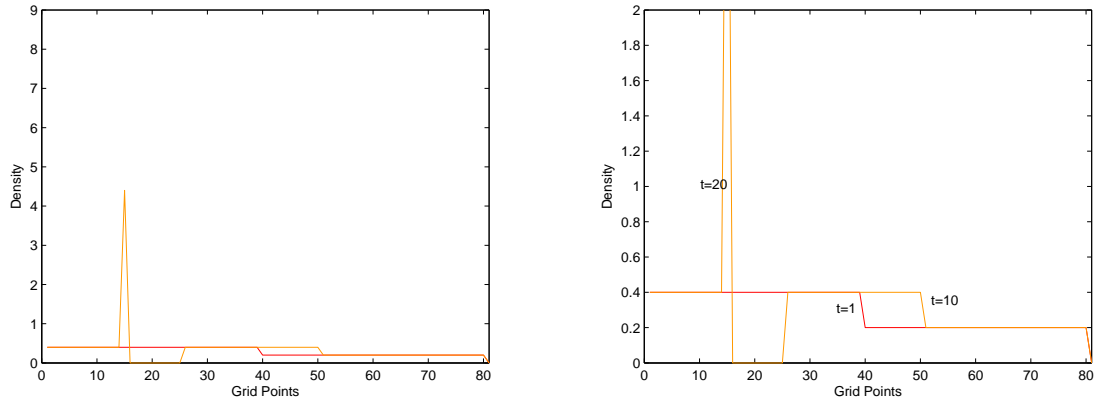


Figure 6.7: Packet Density Scenario, Time: 0,10,20

Figures 6.7, 6.8, 6.9, and 6.10 clearly demonstrate what is happening in this example. First we see the density is growing at the inner grid point  $x = 15$

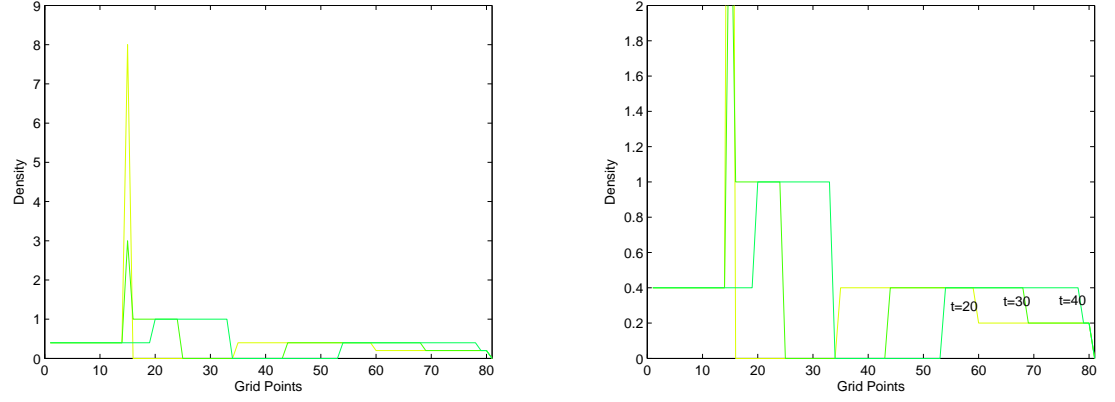


Figure 6.8: Packet Density Scenario, Time: 20,30,40

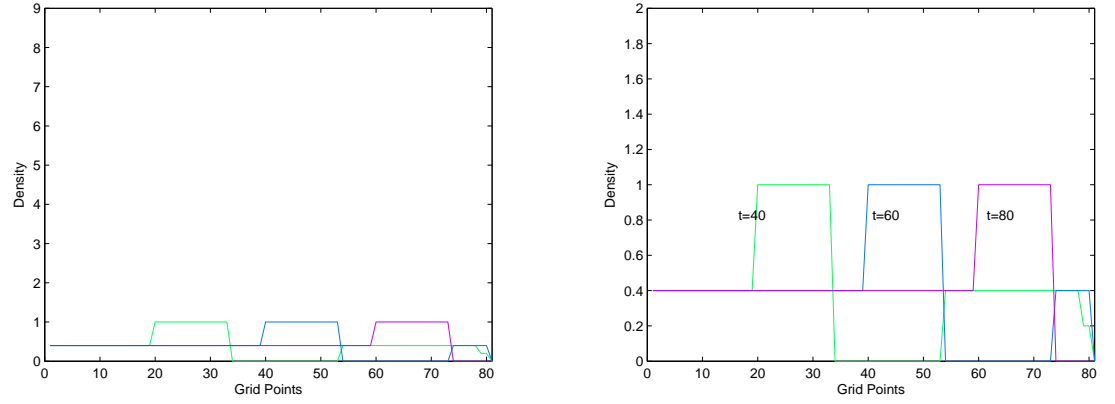


Figure 6.9: Packet Density Scenario, Time: 40,60,80

while it is down for a period of time, which is clearly demonstrated in the figures. When the grid point  $x = 15$  returns to fully functioning the flow recovers quickly.



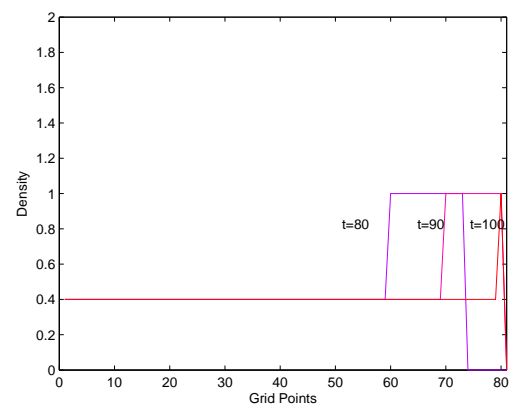
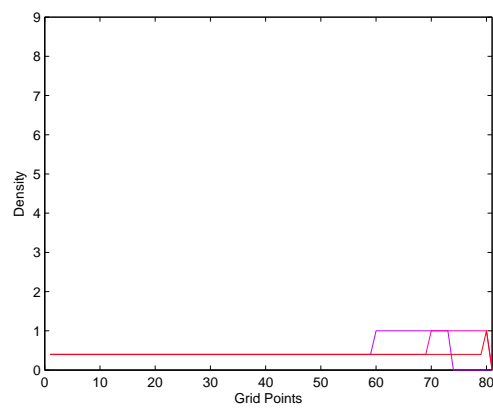


Figure 6.10: Packet Density Scenario, Time: 80,90,100

## 7 Conclusion

The goal was to derive a model for the fluid like aspects of flow in a communication network, and to study the predictions of such a model under conditions of high flow and interruptions. The model derived in this research modeled the behavior of nodes representing routers forwarding packets along links to other routers. A model for the behavior of a single buffer operating a first in, first out queuing protocol using a route matrix. Note that this matrix appears in the flux terms. This requires that each packet in the network is also identified by its destination. As packets flowed through a network from a source to their destination, along the way they are buffered at next appropriate routers along their paths. Counting the number of packets that enter and exit the network at a router, as well as the number of packets that flowed to and from led to the discrete “conservation” of packets equations. In order to view this model as a fluid flow model, spatial location of the routers were defined. Defining the spatial location of routers led to a continuum description of the other variables that describe network flow in this model, namely density, flux etc.

The communication network model derived in this report describes the flow of packets with a continuity equation, giving the density of packets as a function of time and space. The concept of momentum balance, while not finished in this work, can be applied to this model. The analogy with  $\partial_t v + v \cdot \nabla v = -\nabla \omega$  [25] is clear; however, it is less clear what role stress and/or pressure have for communication network flows.

We also explored aspects of the analogy of network traffic flow with cellular automata models. Cellular Automata models attempt to simplify large physical systems by allowing space and time to be discrete and the physical quantities to take only a finite set of values. This approach describes gross flow patterns resulting from packet dynamics.

This communication network model consists of a lattice comprised of nodes and links. The particles on the lattice represent the packets in the internet. Interaction of the packets takes place through the logic implemented by the routers. Each link is a connection between two nodes and packets will travel along predetermined links to get to their destinations. The rules governing the interaction of packets at nodes on the lattice in this model are those implied by equation (5.16), so that “collisions” will be represented by changes in directions implied by the route matrix.

This work studied one-dimensional numerical models applied to various scenarios to examine the behavior of packet flows. Although some interesting behaviors have been found with the one-dimensional model there are still many things that can be studied. For instance, instead of the buffers having infinite capacity, the model could assume that they are finite and when the densities reaches some maximum then packets are lost. Furthermore, a realistic model might assume that when a buffer is full then routers from upstream cannot send packets. This then would represent a more believable backup in the system.

All models simulated the flow for an interval  $0 \leq x \leq 80$  with  $0 \leq t \leq 100$ . In that domain the models demonstrated interesting and physically relevant

behavior. In the first example we focused on flows without saturation, using these calculations to make sure the model performed properly.

The second example demonstrated the flow with saturation. These simulations demonstrate that the maximum flux model seems to behave in a reasonable manner. In the last example we studied the effects of interruptions. Indeed we created network flow model to study the effects of interruptions in a network. In these one-dimensional models, interruptions had a predictable effect on the flow, causing backups in the system. However, backups occurred, not in the overall system but at nodes just upstream of where the interruptions took place. If the density was large enough that flow reached the link packet transport capacity without sending all the packets, a backup occurred and grew linearly without bound for the time while the density exceeded what could be sent. After the saturation conditions are removed the spike decays. Moreover, for interior sources, the combination of low flux and upstream input could also lead to growth at the source grid point.

The work (i) derived a valid model for a discrete buffer-link model of communication network flow, (ii) demonstrated how the FIFO algorithm is implemented in such a model, (iii) derived a continuum model, and (iv) provided some solutions to the model demonstrating how it can describe network saturation and interruptions.

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